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Percolation of the Minority Spins in High-Dimensional Ising Models

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We present some new results on the region in the β–h plane where the + spins percolate for the nearest neighbor Ising model. In particular, it is shown that in high enough dimensions d there is percolation of the minority spins at inverse temperatures $\beta < \beta_+$ with some $\beta_+ > \beta_c$, for which $\beta_+/\beta_c \geqslant \frac{1}{2}\log(cd)$, c a constant.

KEY WORDS: Correlated percolation; Ising model; Peierls argument; minority spins.

1. INTRODUCTION

We present some results on the shape and size of the percolative region $\mathscr{P}$ of the nearest neighbor ferromagnetic Ising model on $\mathbb{Z}^d$. The percolation properties of the connected clusters of (say) the + spins in equilibrium states of these models have been studied because of interest in the properties of correlated percolation (see Refs. 1 and 2) and because of some questions related to the ferromagnetic phase transition.

The system's parameters are defined here so that

$$-\beta H = \beta \sum_{\{x,y\}} \sigma_x \sigma_y + h \sum_x \sigma_x \tag{1}$$

i.e., β is the inverse temperature and h is the external field in units of β, with β_c denoting the inverse temperature above which there is spontaneous

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magnetization. $\mathscr{P}$ is the region in the (β, h) plane where the $+$ spins percolate.

One of our results is that, in large enough dimensions, $\mathscr{P}$ contains a region where $h<0$ and $\beta>\beta_c$, and in particular that the $+$ spins percolate in the $-$ phase for $\beta \in (\beta_c, \beta_+)$, $\beta_+ > \beta_c$ (see Fig. 2). Such a statement is expected to be true for all $d \geqslant 3$, but we cannot prove it. Nevertheless, our lower bounds on the persistence of percolation along the $h=0^-$ line shed some light on the inadequacy in high dimensions of the standard Peierls argument as a tool for bounds on β_c. The results are based on simple applications of FKG-type domination principles and of bounds proven by Men'shikov and by Bramson[3] on the critical densities $p_c(d)$ for independent site percolation models.

Before we turn to a more complete statement of the results (Section 2), let us recall some facts (in reading the following statements it may be useful to consult Figs. 1 and 2). One should note here that there is a qualitative difference between $d=2$ and higher dimensions. This is first seen by considering the $\beta=0$ line, which corresponds to independent site percolation models. For $d \geqslant 3$, numerical calculations[4] indicate that percolation already occurs at negative values of h; specifically, at $\beta=0$ and $h=0$ both $+$ spins and $-$ spins percolate. That situation is not possible in two dimensions, for which the Harris argument[5] and its extension to the interacting systems considered here[1] show that [in the extremal Gibbs states of (1)] there cannot be simultaneous percolation of two opposite spin types. Thus, for $d=2$ it is proved that there is no $+$ percolation for $h<0$ at any $\beta \geqslant 0$ and that at $h=0$ the onset of percolation coincides with the occurrence of long-range order and symmetry breaking; i.e., at $h=0$ the $+$ spins do not percolate if $\beta \leqslant \beta_c$, and for $\beta > \beta_c$ within the $-$ phase (see Fig. 1). Statements known to apply in any dimension $d \geqslant 2$ are that (1) the

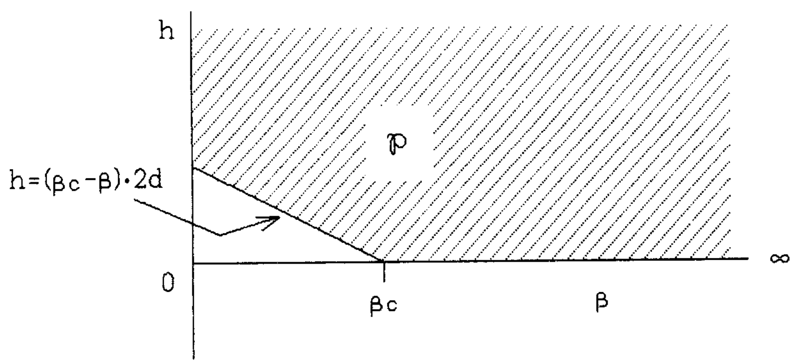

Fig. 1. Subset of $\mathscr{P}$ given by (6).

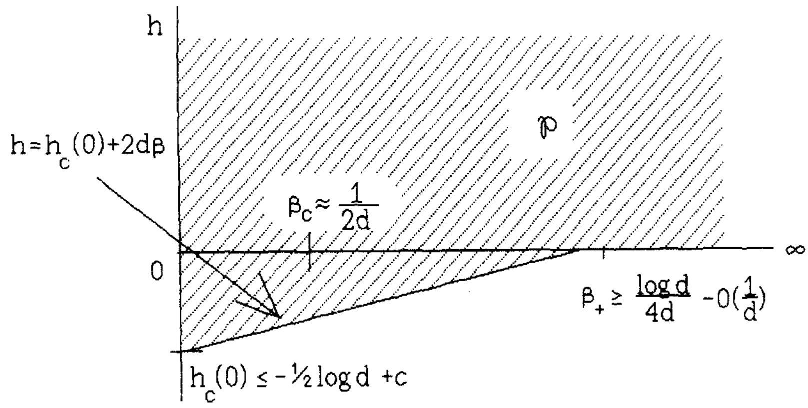

Fig. 2. Subset of $\mathscr{P}$ given by (7) and (11) for large d.

$+$ spins percolate at $h=0$ in the $+$ phase for all $\beta > \beta_c$,[1] (2) for each β there is $+$ percolation at h large enough, and (3) there is β_+, $\beta_c \leqslant \beta_+ < \infty$, so that for $\beta > \beta_+$ there is only $-$ percolation for $h \leqslant 0^-$, i.e., at negative h or in the $-$ phase at $h=0$.

The last assertion is the case addressed by the original Peierls argument,[7] which excludes the possibility of large contours, and in particular the boundary of the infinite $+$ cluster in the $-$ phase. In fact, while that argument is usually used for a bound on β_c, it really provides an upper bound on both β_c and β_+. As we shall see, in high dimensions $\beta_c/\beta_+ \ll 1$. This observation sheds some light on the disparity, as $d \to \infty$, between the behavior of β_c and that of the upper bound obtained by studying the Peierls' contours (see Section 3).

2. RESULTS

We define $\mu(\beta, h)$ as the (unique, for $h \neq 0$) Gibbs state corresponding to (1), with $\mu(\beta, 0^+)$ [resp. $\mu(\beta, 0^-)$] denoting the $+$ state (resp. the $-$ state) for $\beta > \beta_c$. In general, we regard the ray $(h=0, \beta > \beta_c)$ in the (β, h) plane as doubly covered, and write $h \geqslant 0$ to mean $h > 0$ or $h = 0^+$. Let $P_\infty^+(\beta, h)$ be the probability, in $\mu(\beta, h)$, that there exists an infinite connected $+$ cluster. Because this is a tail event and $\mu(\beta, h)$ are extremal Gibbs states, $P_\infty^+(\beta, h) = 0$ or 1.

The percolative regime can be alternatively described as $\mathscr{P} = \{(\beta, h) \mid P_\infty^+(\beta, h) = 1\}$, or as the region where there is a nonzero probability that the origin belongs to an infinite cluster of $+$ spins.

For $\beta=0$, $\mu(\beta, h)$ is just a Bernoulli measure (independent site percolation) with

$$\text{Prob}(\sigma_x = +1) = \tfrac{1}{2}(1 + \langle \sigma_x \rangle_h) = [1 + \exp(-2h)]^{-1} \tag{2}$$

The independent spin measure is trivially monotone "increasing" in h, insofar as the sets of + spins are considered. Moreover, for all $\beta > 0$ the measures $\mu(\beta, h)$ satisfy the conditions for the Fortuin–Kasteleyn–Ginibre[8] inequality, which implies that $P_\infty^+(\beta, h)$ is a monotone non-decreasing function of h. Hence for any dimension there is a function $h_c(\beta)$ such that

$$P_\infty^+(\beta, h) = \begin{cases} 1 & \text{if} \quad h > h_c(\beta) \\ 0 & \text{if} \quad h < h_c(\beta) \end{cases} \tag{3}$$

We now supplement the results described in Section 1 with the following observations, of which the first is based on a more refined application of the FKG inequality.

1. The function $h_c(\beta)$ (which we find to be sometimes increasing and at other times decreasing in β) satisfies

$$|h_c(\beta_1) - h_c(\beta_2)| \leqslant 2d\, |\beta_1 - \beta_2| \tag{4}$$

Equivalently: if there is percolation for some (β', h'), then there is percolation within the cone

$$\{(\beta, h) \,|\, h \geqslant h' + 2d\, |\beta - \beta'|\} \tag{5}$$

This fact follows easily, by the argument of Ref. 9, from the observation that the functions $g_{x,y}(\{\sigma\}) = \sigma_x + \sigma_y \pm \sigma_x \sigma_y$ and the event "the + spins percolate" are increasing in the FKG sense.

The following two statements are obtained by combining the above principle with some auxiliary information.

2. The + spins percolate for

$$h > 2d(\beta_c - \beta) \tag{6}$$

(see Fig. 1) and for

$$h > h_c(0) + 2d\beta \tag{7}$$

(see Fig. 2), where $h_c(0)$ is related to the critical density for independent site percolation [see (2)] by $p_c = \{1 + \exp[-2h_c(0)]\}^{-1}$.

These statements, which can be read as upper bounds on h_c, improve the previous results for nonzero h (see Fig. 2 in Higuchi,[2] where our h is called βh). Condition (6) is a direct consequence of (4) or (5) and the fact that for all $\varepsilon > 0$ there is percolation at $(\beta', h') = (\beta_c + \varepsilon, h = 0^+)$[6] [which can be restated as $h_c(\beta_c) \leqslant 0$], and (7) is obtained by taking $(\beta', h') = (0, h_c(0) + \varepsilon)$. Let us add that (7) has an alternative derivation by a much more general argument, which is based on the fact that any measure "dominates"—in the FKG sense, the independent measure μ that matches the worst conditional probability:

$$\mu(\{\sigma_x = +1\}) = \inf_{\{\tau\}} \text{Prob}(\{\sigma_x = +1\} \mid \sigma_y = \tau_y \text{ for all } y \neq x) \tag{8}$$

3. As in the introduction, let β_+ be the infimum of β for which only the $-$ spins percolate at $h = 0^-$:

$$\beta_+ = \inf\{\beta \geqslant 0 \mid P_\infty^+(\beta, 0^-) = 0, P_\infty^-(\beta, 0^-) > 0 \text{ (i.e.,} = 1)\} \tag{9}$$

The bound (7) yields

$$\beta_+ \geqslant -h_c(0)/(2d) \tag{10}$$

Combining (10) with the bound $p_c \leqslant \hat{c}/d$ of Ref. 3, we get

$$-h_c(0) \geqslant \tfrac{1}{2} \log d - c \tag{11a}$$

and

$$\beta_+ \geqslant \frac{\log d}{4d} - O\left(\frac{1}{d}\right) \tag{11b}$$

Since it follows from the infrared bounds[10] that

$$\beta_c \leqslant [2(d-1)]^{-1} \qquad \text{for} \quad d \geqslant 4 \tag{12}$$

[with $\beta_c(2d-1) \to 1$ as $d \to \infty$] we get a proof that for d large

$$\beta_+/\beta_c \geqslant \tfrac{1}{2} \log d - c' \gg 1 \tag{13}$$

3. CONCLUSIONS FOR HYPERSURFACE ENTROPY

By the remarks on the Peierls contour argument made in the introduction, the inequalities (11)–(13) show that in its usual form it does not extend as close to the true value of β_c as the infrared bounds do. Related to the above observation is a lower bound on the entropy of closed hypersurfaces.

The basic estimate of Peierls is that the probability of having a + spin at the origin, for a state obtained with − boundary condition, is less than

$$\sum_{\gamma} \exp(-2\beta\, |\gamma|) \tag{14}$$

where the sum runs over all contours (which are the outer boundaries of connected sets) enclosing the origin, and $|\gamma|$ is the (hyper-) area of γ [i.e., the cardinality of the set of $(d-1)$-dimensional unit cubes forming γ] While it is quite clear that when the sum in (14) is less than 1/2 there is long-range order, it is also true that the mere convergence of this sum has such an implication.

Indeed, if (14) converges, there are no infinite contours, i.e., no infinite clusters of + spins in the − phase. Moreover, by the Borel–Cantelli lemma (the easy part), we know that if (14) converges, there are only a finite number of contours surrounding the origin. This in turn implies that there is an infinite cluster of − spins (for − b.c.). and thus a breakdown of symmetry, which (by an FKG argument) can occur only if there are two different Gibbs states and the spontaneous magnetization does not vanish.

Thus, with

$$\begin{aligned} K(n) &= \#\{\gamma \mid \gamma \text{ encloses } 0 \text{ and } |\gamma| = n\} \\ s(d) &= \lim_{n \to \infty} n^{-1} \log K(n) \end{aligned} \tag{15}$$

(which exists by a subadditivity argument) we define the Peierls inverse temperature by

$$\beta_{\mathrm{P}} = \tfrac{1}{2} s(d) \tag{16}$$

It is clear that $\beta_{\mathrm{P}} = \inf\{\beta \mid (14) \text{ converges}\}$, and hence by the above argument

$$\beta_{\mathrm{P}} \geqslant \beta_{+} \geqslant \beta_{c} \tag{17}$$

While the best upper bound we are aware of for $s(d)$ is $s(d) \leqslant \text{const}$[11] the inequalities (11) and (17) yield

$$s(d) \geqslant \frac{\log d - c}{2d} \tag{18}$$

which we expect to be closer to the correct asymptotic behavior as $d \to \infty$. Precise estimates of $s(d)$ for d large will have to address the reduction in the entropy of contours (i.e., their number) caused by the fact that they are closed (in addition to being connected). Anyway, we saw that even an optimal estimate of $s(d)$ would not carry us as close to β_c, for d large, as the infrared bounds do.

Finally, we remark that for β in the interval $\beta_c < \beta < \beta_+$ one cannot use the standard contours as a good notion of an interface between the + and the − phases in a situation like the Dobrushin ± boundary conditions.[12] Further remarks on this point (which is relevant for the study of the "roughening transition") can be found in Refs. 13 and 14. Some alternative methods for studying interfaces can be based on arguments in Ref. 15.

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